

I think there should be a term before the two KL divergence term.

Conjecture 1: Given Monge's formulation:

(1)   
where for some , and ,

we claim: the optimal is a 1-1 mapping.

It is equivalent to Corollary 3.3.

Proof: Given which is not 1-1 mapping. We reindex and such that:

1.

.

2. where and .

3 .

We define such that for .   
Let and denote the optimal joint measure induced by and respectively.

For convenience, let denote the mass of point and similarly . Now we analyze for .

The object function (1) becomes

where is a constant which is independent to . Let denote the function above.

Take derivative:



1. There exists such that for each fixed .   
In fact, we have close form:



. (1.1)

2. is convex with respect to each .

Why? .

Then the optimal satisfies .

Similarly, we have .



That is . (1.2)

By (1.1)(1.2), we have



Then . (1.3)



For , since , then .

Let . We need to compare .

.



by plugging in (1.1) (1.2)

. By (1.3)

Does it violate to the conjecture 1?

If we transfer and then destroy it.

Original: Say .

New:

Then difference is

Particular example:   
Consider the example: . .



The optimal 1-1 mapping, dented as is defined as .



We define by



Let denote the optimal joint distribution induced by respectively.



. (0 transportation cost and is the cost for destroying and creating .)



Now we define



We have

.

Note, given 1-1 mapping , by (1.1), we can derive the closed form of

. Suppose .

That is, , .

Then we have



.

Compare to the result

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Description automatically generated with medium confidence

I fee there is a typo at here. Consider . As , the distance will not be a continuous function (with respect to ).

Conjecture 2: The solution of unbalanced OT problem/ Logarithmic Entropy-Transport/ Hellinger–Kantorovich OT problem:

where is induced by a mapping .

Text

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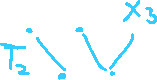
The related theorem is the following:



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Consider the following example:



We first consider the following two mappings:

. We aim to show are not optimal mappings.

We have: .

Let denoted the optimal joint distribution induced by respectively. We have:

, indeed, while .

Then by Theorem 6.6 (i), both can not be optimal mappings.

Similarly, we consider the following mappings .



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We have . But are pair-wisely different.

Then can not be the optimal mappings.

We do not have other choices of (monotonic increasing) mapping .

So, does it violate the conjecture 2?